ON THE PAPER "PENETRATION OF A WEDGE INTO COMPRESSIBLE HALF-SPACE" BY Z.N. DOBROVOL'SKAIA

(O RABOTE Z.N. DOBROVOL'SKOI "PRONIKANIE KLINA V SZHIMAEMOE POLUPROSTRANSTVO")

PMM Vol.26, No.3, 1962, pp. 557-558

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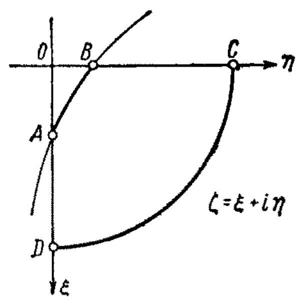
(Received September 30, 1961)

The problem investigated in [1] considers the immersion, with constant velocity, of a wedge with arbitrary opening angle 2α into half-space, filled with a compressible fluid, in the absence of gravity forces. For the case when the quantity $M = v/a_0$ is small (v is velocity of the wedge and a_0 is the unperturbed sound velocity in the medium), the problem may be linearized. The author solves it for this case. Thanks to selfsimilarity of the motion, the problem reduces to a boundary value problem for Laplace's equation in the plane of the variable $\zeta = \varepsilon \exp(i\theta)$ (using the notations of [1]) for the region $ABCD$ in the shape of a quarter of a unit circle with a discarded angle, (see figure) and with homogeneous boundary conditions [1]. To solve this problem the author maps the region $ABCD$ on a half-plane, whereby, in view of the difficulty of constructing an exact mapping function (because of the presence of the curvilinear segment of the boundary AB) a function was constructed which maps on the half-plane not the region $ABCD$, but a region close to it, which was obtained by replacing the segment of the boundary AB by a curve close to AB, having an analytic representation which was convenient for an explicit representation of the mapping function. Subsequently a formula was evolved which gives a solution of Hilbert's problem for the half-plane. The resulting relations are complicated and do not give an effective solution of the problem. Only in the case of immersion of a thin wedge ($\alpha \to 0$) is it possible to represent the solution in terms of elementary functions and it coincides then with the solution obtained earlier by Sagomonian [2].



The solution of the problem given in the paper by Dobrovol'skaia [1] is incorrect. To construct a solution of the linear problem considered, there is no need for elaborate construction with approximate conformable mapping, which constitutes the basic contents of [1]. A correct solution to the problem may be obtained by elementary means by exhibiting explicitly the assumptions of the procedure leading to a linear formulation of the problem.

We note that the linearization with respect to the small parameter M, which is carried out in [1], is admissible only at distances from the wedge which are considerable as compared to the dimension of the immersed part of the wedge.

Indeed, the region embraced by the motion at the instant t is bounded by the wetted surface of the wedge, the free surface and the semi-circle of radius $a_0 t$ with center at the point of entry of the wedge. The size of the immersed part of the wedge will thereby be of the order vt, and of the same order will also be the size of the part of the free surface located near the wedge which underwent an essential deviation from the initial rectilinear position. But $M = v/a_0$ is a quantity which is small as compared to unity.

This means that the dimensions of the region of the fluid in motion in which nonlinear effects are pronounced will be small as compared to the characteristic dimension of the total region of motion where the disturbance will be small and may be described by linear relations.

Therefore, in solving the linear problem, it should be assumed that the solution of this problem is determined everywhere within the semicircle of radius $a_0 t$ with the center at the point of entry of the wedge into the fluid, with the exception of this point itself at which the solution possesses a singularity. In the vicinity of this point the solution will have an asymptotic which must be used to match the solution of the linearized problem with the solution of the nonlinear problem at distances which are considerable as compared to vt, which is the size of the region where nonlinearity is essential, but still small as compared to $a_0 t$, the size of the region where the motion is linear. This matching will relate the arbitrary parameter, contained in the asymptotic of the linear solution, with the angle α.

In solving the linear problem in no case should one require that it satisfies exactly the flow condition past the wedge, i.e. the exact boundary condition along the small part of the boundary AB, because in the vicinity of AB the linear solution is not valid: the linear solution should be, I repeat, related to the nonlinear part only by means of matching of their asymptotic representations. But the author of [1] proceeds precisely in this inadmissible manner.

Let us present the solution to the problem. A correct formulation of the linear problem on the basis of what was said above reduced to the following. One has to find for the quarter of the unit circle in the plane $\zeta = \varepsilon \exp(i\theta)$ a harmonic function p (pressure) with the boundary conditions (figure)

$$p|_{OC} = 0, \qquad \frac{\partial p}{\partial \eta}\Big|_{OD} = 0, \qquad p|_{CD} = 0 \tag{1}$$

The function p must be regular everywhere in the region OCD, including the boundary, except at the point O where it must have a singularity of a definite type. The solution to this problem, for instance by means of mapping of the region OCD of the semi-plane and application of Keldysh-Sedov formula [3] is written down immediately. It is of the form

$$p = \rho_0 v^2 c(M, \alpha)\left(\frac{1}{\varepsilon} - \varepsilon\right)\cos\theta \tag{2}$$

The dimensionless constant C depends on the wedge angle α and the number M.

In order to find this dependence we must solve the nonlinear problem on the immersion of the wedge into the incompressible fluid (this problem has not yet been solved) and find the asymptotic of this solution at distances far away from the wedge which gives the following formula for the pressure

$$p = \rho_0 v^2 c_1(\alpha)\frac{vt}{r}\cos\theta \tag{3}$$

because this asymptotic will be nothing else but flow from a planė dipole. The requirement of coincidence of the right parts of (2) and (3) with the condition that

$$\varepsilon \ll 1, \qquad vt/r \ll 1, \qquad r/a_0 t = 2\varepsilon$$

(the latter follows from the Chaplygin transformation for $\varepsilon \to 0$) will determine $C(M, \alpha)$

$$C(M, \alpha) = \frac{1}{2} M C_1(\alpha) \tag{4}$$

Here $C_1(\alpha)$ is a known quantity.

It follows from Formula (4) that the smallness of motion in the linearized problem consists in that the pressure p is proportional to M, the small parameter of the problem.

Finally, it should be pointed out that for $\alpha \to 0$ the solution in [1] becomes correct and this is associated with the fact that in this case the linearization is not with respect to the parameter M, but with respect to the angle α. The parameter M in this case may acquire any positive value and the linearization for a small α will still be admissible. The boundary condition on the wedge in this case should be satisfied exactly by the solution of the linearized problem and not by the "dipole" asymptotic as this was done above for $M \ll 1$. Thus the solution of [1] becomes correct only for small α (for reasons which are not indicated in [1]).

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Translated by G.H.